Chapter 1
POLYNOMIAL FUNCTIONS

Have you ever wondered how computer graphics software is able to so quickly draw the smooth, life-like faces that we see in video games and animated movies? Or how in architectural projects builders compensate for the fact that a horizontal beam, fixed in position at both ends, will bend under its own weight? Can you imagine how computers mould automotive body panels? Believe it or not, all three tasks are possible thanks to polynomials! Polynomials are composed by applying addition, subtraction, and multiplication to numbers and variables. The information needed to perform certain tasks like the ones listed above is reduced to the polynomial segments between key points. Much like words in language, polynomials are the vocabulary of algebra, and, as such, they are used in a wide variety of applications by designers, engineers, and others. Calculus, the study of motion and rates of change, requires a clear understanding of polynomials, so we'll begin our study there.

CHAPTER EXPECTATIONS In this chapter, you will

- determine properties of the graphs of polynomial functions, **Section 1.1**
- sketch the graph of a polynomial function, **Section 1.1**
- describe the nature of change in polynomial functions, **Section 1.2**
- determine an equation to represent a given graph of a polynomial function, **Career Link**
- understand the Remainder and Factor Theorems, **Section 1.3, 1.4**

Review of Prerequisite Skills

Before beginning your study of Polynomial Functions, you may wish to review the following factoring methods that you learned in previous courses.

Common Factor

- $4x^2 - 8x = 4x(x - 2)$

Grouping

- By grouping terms together it is often possible to factor the grouped terms.
  Factor fully $\quad ax + cx + ay + cy = (ax + cx) + (ay + cy)$
  $$= x(a + c) + y(a + c)$$
  $$= (a + c)(x + y)$$

Trinomial Factoring

- Factor fully $3x^2 - 7x + 4$.

Solution 1 (by decomposition)

$$\begin{aligned} 3x^2 - 7x + 4 &= 3x^2 - 3x - 4x + 4 \\ &= 3x(x - 1) - 4(x - 1) \\ &= (x - 1)(3x - 4) \end{aligned}$$

Solution 2 (by inspection)

$$3x^2 - 7x + 4 = (x - 1)(3x - 4)$$

Factor $12x^2 - x - 20$.

Solution

Create a chart using factors of 12 and –20.

12	6	4	$\cancel{20}$	$-\cancel{20}$	$\cancel{10}$	$-\cancel{10}$	5	−5	1	−1	$\cancel{2}$	$-\cancel{2}$	$\cancel{4}$	$-\cancel{4}$
1	2	3	$-\cancel{1}$	$\cancel{1}$	$-\cancel{2}$	$\cancel{2}$	−4	4	−20	20	$-\cancel{10}$	$\cancel{10}$	$-\cancel{5}$	$\cancel{5}$

Notice that what looks like a lot of work can be greatly simplified when numbers in the upper right that have common factors with 12, 6, and 4 are crossed out.

The reduced chart is

12	6	4	5	−5	1	−1
1	2	3	−4	4	−20	20

From the numbers that remain, we see that $4 \times (-4) = -16$, and $3 \times 5 = 15$ gives $-16 + 15 = -1$. Therefore, $12x^2 - x - 20 = (4x + 5)(3x - 4)$.

Difference of Squares

- Because $(a + b)(a - b) = a^2 - b^2$, it is always possible to factor the difference between two perfect squares.
  $16x^2 - 81 = (4x + 9)(4x - 9)$

Special Cases

- Sometimes by grouping terms, the difference between squares can be created.

$$\begin{aligned} a^2 - p^2 + 1 + 2a &= (a^2 + 2a + 1) - p^2 \\ &= (a + 1)^2 - p^2 \\ &= [(a + 1) + p][(a + 1) - p] \\ &= (a + 1 + p)(a + 1 - p) \end{aligned}$$

Exercise

1. Factor fully.
   a. $p^2 + 2pr + r^2$
   b. $16n^2 + 8n + 1$
   c. $9u^2 + 30u + 25$
   d. $v^2 + 4v + 3$
   e. $2w^2 + 3w + 1$
   f. $3k^2 + 7k + 2$
   g. $7y^2 + 15y + 2$
   h. $5x^2 - 16x + 3$
   i. $3v^2 - 11v - 10$

2. Factor fully.
   a. $25x^2 - y^2$
   b. $m^2 - p^2$
   c. $1 - 16r^2$
   d. $49m^2 - 64$
   e. $p^2r^2 - 100x^2$
   f. $3 - 48y^2$
   g. $(x + n)^2 - 9$
   h. $49u^2 - (x - y)^2$
   i. $x^4 - 16$

3. Factor fully.
   a. $kx + px - ky - py$
   b. $fx - gy + gx - fy$
   c. $h^3 + h^2 + h + 1$
   d. $x - d + (x - d)^2$
   e. $4y^2 + 4yz + z^2 - 1$
   f. $x^2 - y^2 + z^2 - 2xz$

4. Factor fully.

a. $4x^2 + 2x - 6$
b. $28s^2 + 8st - 20t^2$
c. $y^2 - (r - n)^2$
d. $8 + 24m - 80m^2$
e. $6x^2 - 13x + 6$
f. $y^3 + y^2 - 5y - 5$
g. $60y^2 - 10y - 120$
h. $10x^2 + 38x + 20$
i. $27x^2 - 48$

5. Factor fully.

a. $36(2x - y)^2 - 25(u - 2y)^2$
b. $g(1 - x) - gx + gx^2$
c. $y^5 - y^4 + y^3 - y^2 + y - 1$
d. $n^4 + 2n^2w^2 + w^4$
e. $9(x + 2y + z)^2 - 16(x - 2y + z)^2$
f. $8u^2(u + 1) + 2u(u + 1) - 3(u + 1)$
g. $p^2 - 2p + 1 - y^2 - 2yz - z^2$
h. $9y^4 + 12y^2 + 4$
i. $abx^2 + (an + bm)x + mn$
j. $x^2 + 2 + \frac{1}{x^2}$

Career Link investigate

CHAPTER 1: MODELLING WATER DEMAND

Imagine if you woke up one morning looking forward to a shower only to have your mom tell you the local water utility ran out of water because they made a mistake in predicting demand. That does not happen, in part, because water utilities develop reliable mathematical models that accurately predict water demand. Of particular use in mathematical modelling are the polynomial functions that you will investigate in this chapter. You are already familiar with two classes of polynomials: the linear ($y = mx + b$) and the quadratic ($y = ax^2 + bx + c$). You can find polynomial mathematical models in a multitude of places, from computers (e.g., Internet encryption), to business (e.g., the mathematics of investment), to science (e.g., population dynamics of wildlife).

Case Study — Municipal Engineer/Technologist

Civil Engineers and Technologists frequently model the relationship between municipal water demand and time of day to ensure that water supply meets demand plus a factor of safety for fire flows. Water demand data for a city with a population of 150 000 is presented in the table below.

Water Demand for Blueborough, Ontario

Time of Day	t (in hours)	Water Demand (in cubic metres per hour)
13:00	1	5103
14:00	2	4968
15:00	3	5643
16:00	4	7128
17:00	5	8775
18:00	6	9288
19:00	7	6723

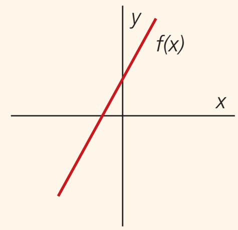
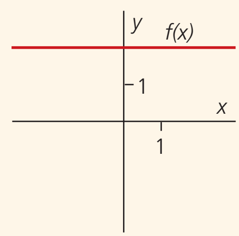
DISCUSSION QUESTIONS

1. Plot a rough sketch of the data in the table above. What kind of relationship, if any, does the data show? Remember that you have been investigating linear, quadratic, rational, and periodic functions. Does the hour-to-hour trend in the data make sense? Explain.
2. Sketch the water demand over a 24-h period for your community. Use an average daily demand of 600 L per capita and a peak hourly flow of about 2.5 times the average hourly flow. Explain the peaks and valleys.
3. Find out how much water costs in your community and estimate the cost per hour of operating your community's water distribution system at the peak flow rate determined in Question 1.

At the end of this chapter you will develop and utilize a mathematical model for the data presented in this case study.

Section 1.1 — Graphs of Polynomial Functions

The graph of a linear function of the form $f(x) = ax + b$ has either one x-intercept or no x-intercepts.

Function	Graph	Number of x-intercepts
$f(x) = 2x + 1$		1
$f(x) = 2$		No x-intercepts

By graphing a quadratic function of the form $f(x) = ax^2 + bx + c$, $a \neq 0$, we can determine the number of x-intercepts. Each x-intercept indicates a real root of the corresponding quadratic equation.

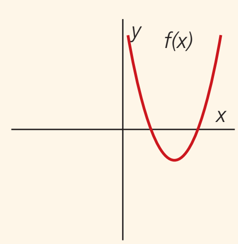
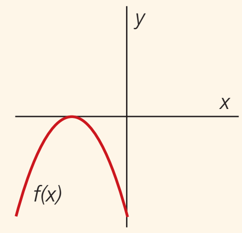
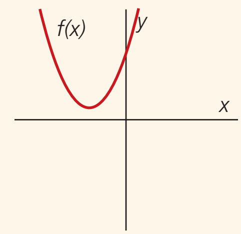
Function	Graph	Number of x-intercepts
$f(x) = x^2 - 7x + 10$		2

chart continued

$f(x) = -x^2 - 6x - 9$		1 When a curve touches the x-axis, there are two equal roots for the corresponding
$f(x) = 2x^2 + 3x + 4$		0 There are no real roots.

INVESTIGATION 1: CUBIC FUNCTIONS

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1. Use a graphing calculator or a computer to graph each of the following cubic functions. Sketch each of the graphs in your notebook so that you can make observations about the shapes of the graphs and list the number of x-intercepts.

   a. $y = x^3$
   b. $y = x^3 + 2x$
   c. $y = x^3 - 2x^2$
   d. $y = 2x^3 - 3$
   e. $y = 2x^3 - 5x^2 - 8x + 12$
   f. $y = x^3 - 3x + 2$
   g. $y = 4x^3 - 16x^2 + 13x - 3$
   h. $y = x^3 - 5x^2 + 2x + 8$
   i. $y = (x - 2)(x + 1)(3x - 1)$

2. From your observations, list the possible numbers of real roots for a cubic equation.

3. a. Explain how you would graph the cubic function $y = (x - 2)(x + 3)(x - 4)$ without using a graphing calculator.

   b. Draw a sketch of the function in part **a.**

4. Sketch two possible general shapes for the graph of a cubic function that has a coefficient of x^3 that is positive.

5. For the functions in Question 1, change the coefficient of x^3 from positive to negative and redraw the graphs. For example, $y = x^3 - 2x^2$ changes to $y = -x^3 - 2x^2$. What observation do you make for the general shape of the graph of a cubic function that has a coefficient of x^3 that is negative?

INVESTIGATION 2: QUARTIC FUNCTIONS

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1. Use a graphing calculator or a computer to graph each of the following quartic functions. Sketch each of the graphs in your notebook so that you can make observations about the shapes of the graphs and list the number of x-intercepts.

   a. $y = x^4$ b. $y = x^4 - 4$

   c. $y = x^4 - 3x^3$ d. $y = x^4 - 3x^3 - 12x^2$

   e. $y = x^4 - 3x^3 - 6x^2 + 2x - 3$ f. $y = (x - 1)(x + 2)(x - 3)(2x - 3)$

2. From your observations, list the possible numbers of real roots for a quartic equation.

3. a. Explain how you would graph the quartic function $y = (x + 3)(x - 2)(x + 1)(x + 4)$ without using a graphing calculator.

   b. Draw a sketch of the function in part **a**.

4. Sketch two possible general shapes for the graph of a quartic function that has a coefficient of x^4 that is positive.

5. For the functions in Question 1, change the coefficient of x^4 from positive to negative and redraw the graphs. For example, $y = x^4 - 3x^3$ changes to $y = -x^4 - 3x^3$. What observation do you make for the general shape of the graph of a quartic function that has a coefficient of x^4 that is negative?

INVESTIGATION 3

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1. Use your graphing calculator to graph each of the following:

   a. $y = x(x - 3)^2$ b. $y = (x - 1)(x + 2)(x + 1)^2$

   c. $y = (x + 2)^2(x - 2)^2$

   Based on these graphs, draw a sketch of what you think the graph of $y = (x + 2)(x - 1)^2$ looks like.

2. Use your graphing calculator to graph each of the following:

a. $y = (x - 2)^3$ b. $y = x(x - 3)^3$ c. $y = (x - 1)^2(x + 1)^3$

Based on these graphs, draw a sketch of what you think the graph of $y = (x + 1)(x - 1)^3$ looks like.

Exercise 1.1

Part A

Knowledge/Understanding

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1. Check your conclusions about the shape of the graphs of functions by using your graphing calculator to draw each of the following:

a. $y = x^3 - 12x + 16$
b. $y = x^3 - x^2 - 10x + 15$
c. $y = -2x^3 + 11x + 6$
d. $y = -2x^4 + 3x^3 - 5$
e. $y = (2x + 3)(3x - 1)(x + 2)(x - 3)$
f. $y = (x - 1)(x^2 - 3)(9x^2 - 4)$
g. $y = x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$
h. $y = -x^5 + 4x^3 + x^2 - 3x - 3$

Part B

Application

2. Draw a rough sketch (without using your graphing calculator) of each of the following:

a. $y = (x + 1)(x - 2)$
b. $y = (x + 2)(x - 1)(x + 3)$
c. $y = (x - 2)(x + 3)(x + 1)(x - 4)$
d. $y = (x - 1)(x + 2)^2$

Communication

3. a. Draw as many different shapes as possible of a cubic function.

b. Draw as many different shapes as possible of a quartic function.

Thinking/Inquiry/Problem Solving

4. You have investigated the general shape of the graphs of cubic and quartic functions. Sketch a possible general shape for the graphs of each of the following:

a. A fifth-degree function that has a coefficient of x^5 that is

(i) positive (ii) negative

b. A sixth-degree function that has a coefficient of x^6 that is

(i) positive (ii) negative

Section 1.2 — Polynomial Functions from Data

In earlier courses, you used finite differences as a means of identifying polynomial functions. If we have the right data we can obtain a sequence of first differences, second differences, and so on. The purpose of the investigation in this section is to determine the pattern of finite differences for given polynomials.

The table below lists finite differences for the linear function $f(x) = x$.

x	$f(x)$	$\Delta f(x)$
1	1	$2 - 1 = 1$
2	2	$3 - 2 = 1$
3	3	$4 - 3 = 1$
4	4	$\vdots$
$\vdots$	$\vdots$	$\vdots$
$m - 1$	$m - 1$	$m - (m - 1) = 1$
m	m	$m + 1 - m = 1$
$m + 1$	$m + 1$	

The set of first differences of a linear function is constant.

INVESTIGATION

The purpose of this investigation is to determine the pattern of finite differences for quadratic and cubic functions.

1. For the function $f(x) = x^2$, copy and complete the table below, calculating first differences, second differences, and so on, to determine whether or not the sequence of entries becomes constant.

x	$f(x)$	$\Delta f(x)$ first difference	$\Delta^2 f(x)$* second difference	$\Delta^3 f(x)$ third difference
1				
2				
3				
$\vdots$				
$m - 2$				
$m - 1$				
m				
$m + 1$				
$m + 2$				

*$\Delta^2 f(x)$ means second difference.

2. For the function $f(x) = x^3$, copy and complete the table below, calculating first differences, second differences, and so on, to determine whether or not the sequence of entries becomes constant.

x	$f(x)$	$\Delta f(x)$ first difference	$\Delta^2 f(x)^*$ second difference	$\Delta^3 f(x)$ third difference
1				
2				
3				
⋮				
$m - 2$				
$m - 1$				
m				
$m + 1$				
$m + 2$				

$^*\Delta^2 f(x)$ means second difference.

If the set $\{m - 2, m - 1, m, m + 1, m + 2\}$ describes every set of five consecutive x values, can you make a general statement about the pattern of successive finite differences for polynomial functions?

EXAMPLE

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Given that the points (1,1), (2, −3), (3, 5), (4, 37), (5, 105), and (6, 221) lie on the graph of a polynomial function, determine a possible expression for the function having integer coefficients.

Solution

Input the data in your graphing calculator as follows:

1. Select the STAT function and press ENTER to select EDIT mode.

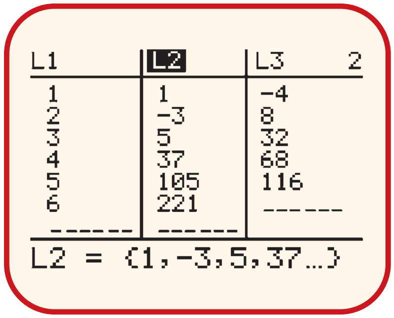
2. In the L_1 column, input 1, 2, 3, 4, 5, 6, and for the L_2 column, input 1, −3, 5, 37, 105, 221.

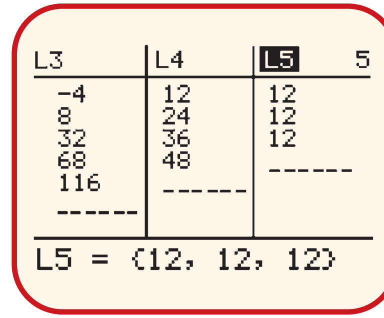
3. Move the cursor to the L_3 column. Select 2nd LIST STAT for the LIST function. Move the cursor to OPS and then select option **7:ΔList(**.

4. Enter L_2 in the **ΔList** (L_2) to obtain the first finite differences for L_2.

5. Move the cursor to the L_4 column. Repeat steps 3 and 4 to obtain the second finite differences for L_3. Note: Enter L_3 in the **ΔList** (L_3).

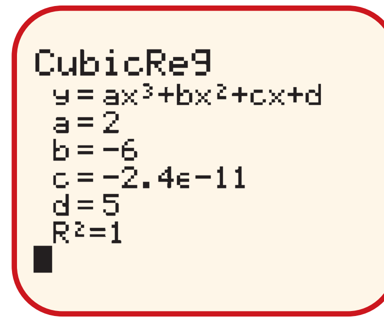
6. Move the cursor to the L_5 column. Repeat steps 3 and 4 to obtain the third finite differences for L_3. *Note*: Enter L_4 in the **ΔList** (L_4).





If the first finite difference is constant, then $f(x)$ is a linear function. If the second finite difference is constant, then $f(x)$ is a quadratic function.

The third finite difference in column L_5 is constant. If $f(x)$ is a polynomial function, then it must be cubic, of the form $f(x) = ax^3 + bx^2 + cx + d$. Use the **CubicReg** function to obtain the following result. The **CubicReg** function is located in the CALC mode on the STAT key.



Note that $c = -2.4 \times 10^{-11}$ is a very small number, so let $c = 0$ and the required result is $f(x) = 2x^3 - 6x^2 + 5$.

A second method, using algebra, is as follows. Let the function be $f(x)$.

Using differences, we obtain the following:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1	−4	12	12
2	−3	8	24	12
3	5	32	36	12
4	37	68	48	
5	105	116		
6	221			

From the data, $\Delta^3 f(x)$ is constant. If $f(x)$ is a polynomial, it must be cubic, therefore $f(x)$ must be of the form $f(x) = ax^3 + bx^2 + cx + d$.

Using the given ordered pairs, we get

$$f(1) = a + b + c + d = 1 \quad ①$$
$$f(2) = 8a + 4b + 2c + d = -3 \quad ②$$
$$f(3) = 27a + 9b + 3c + d = 5 \quad ③$$
$$f(4) = 64a + 16b + 4c + d = 37 \quad ④$$

Solving these equations, we have

② − ① $\quad 7a + 3b + c = -4$ ⑤

③ − ② $\quad 19a + 5b + c = 8$ ⑥

④ − ③ $\quad 37a + 7b + c = 32$ ⑦

⑥ − ⑤ $\quad 12a + 12b = 12$ ⑧

⑦ − ⑥ $\quad 18a + 2b = 24$ ⑨

⑨ − ⑧ $\quad 6a = 12$

$a = 12$

Substituting into ⑧ $\quad 24 + 2b = 12$

$b = -6$

Substituting into ⑤ $\quad 14 - 18 + c = -4$

$c = 0$

Substituting into ① $\quad 2 - 6 + 0 + d = 1$

$d = 5$

Therefore, the function is $f(x) = 2x^3 - 6x^2 + 5$.

Exercise 1.2

Part A

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In each of the following, you are given a set of points that lie on the graph of a function. Determine, if possible, the equation of the polynomial function using a graphing calculator or the algebraic method.

Knowledge/Understanding

1. (1, 0), (2, −2), (3, −2), (4, 0), (5, 4), (6, 10)
2. (1, −1), (2, 2), (3, 5), (4, 8), (5, 11), (6, 14)
3. (1, 4), (2, 15), (3, 30), (4, 49), (5, 72), (6, 99)
4. (1, −9), (2, −10), (3, −7), (4, 0), (5, 11), (6, 26)
5. (1, 12), (2, −10), (3, −18), (4, 0), (5, 56), (6, 162)
6. (1, −34), (2, −42), (3, −38), (4, −16), (5, 30), (6, 106)
7. (1, 10), (2, 0), (3, 0), (4, 16), (5, 54), (6, 120), (7, 220)
8. (1, −4), (2, 0), (3, 30), (4, 98), (5, 216), (6, 396)
9. (1, −2), (2, −4), (3, −6), (4, −8), (5, 14), (6, 108), (7, 346)
10. (1, 1), (2, 2), (3, 4), (4, 8), (5, 16), (6, 32), (7, 64)

Part B

Application

11. The volume, V, of air in the lungs during a 5 s respiratory cycle is given by a cubic function (with time t as the independent variable).

a. The following data was recorded:

t (in seconds)	V (in litres)
1	0.2877
2	0.6554
3	0.8787
4	0.7332

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Determine the cubic function that satisfies this data.

b. Using your graphing calculator, find the maximum volume of air in the lungs during the cycle, and find when during the cycle this maximum occurs.

Thinking/Inquiry/Problem Solving

12. a. The population of a town is given by a polynomial function. Let time, t, be the independent variable, $t = 0$ in 1981, and use the data below to determine the function.

Year	Population
1981	4031
1982	4008
1983	3937
1984	3824
1985	3675
1986	3496

b. The town seemed destined to become a "ghost town" until oil was discovered there and the population started to increase. In what year did this happen?

c. If the function continues to describe the population correctly, what will the population be in 2030?

Section 1.3 — Division of Polynomials

Division of polynomials can be done using a method similar to that used to divide whole numbers. Since division of polynomials cannot be done on all calculators, let's first review the division process in arithmetic.

EXAMPLE 1 Divide 579 by 8.

Solution

$$\begin{array}{r} 72 \\ 8\overline{)579} \\ \underline{56} \\ 19 \\ \underline{16} \\ 3 \end{array}$$

Step 1: Divide 8 into 57, obtaining 7.
Step 2: Multiply 8 by 7, obtaining 56.
Step 3: Subtract 56 from 57, obtaining 1.
Step 4: Bring down the next digit after 57.
Step 5: Repeat steps 1−4 using the new number, 19.
Step 6: Stop when the remainder is less than 8.

We can state the results in the form of the division statement $579 = 8 \times 72 + 3$. Division with polynomials follows the same procedure. When you are performing division, you should write both the divisor and dividend in descending powers of the variable.

EXAMPLE 2 Divide $x^2 - 7x - 10$ by $x + 2$.

Solution

$$\begin{array}{r} x - 9 \\ x+2\overline{)x^2 - 7x - 10} \\ \underline{x^2 + 2x} \\ -9x - 10 \\ \underline{-9x - 18} \\ 8 \end{array}$$

Step 1: Divide first term of the dividend $(x^2 - 7x - 10)$ by the first term of the divisor [i.e., $x^2 \div x = x$].
Step 2: Multiply $(x(x + 2) = x^2 + 2x)$, placing the terms below those in the dividend of the same power.
Step 3: Subtract and bring down the next term.
Step 4: Repeat steps 1−3.
Step 5: Stop when the degree of the remainder is less than that of the divisor.

We can express the results as $x^2 - 7x - 10 = (x + 2)(x - 9) + 8$.
Note: This is of the form dividend = divisor × quotient + remainder or $f(x) = d(x)q(x) + r(x)$.

EXAMPLE 3 Perform the following divisions and express the answers in the form $f(x) = d(x)q(x) + r(x)$.

a. $(2x^3 + 3x^2 - 4x + 3) \div (x + 3)$ b. $(x^3 - x^2 - 4) \div (x - 2)$

Solution

a.

$$
\begin{array}{r|rrrr}
 & 2x^2 & -3x & +5 & \\
\hline
x+3 & 2x^3 & +3x^2 & -4x & +3 \\
 & \underline{2x^3} & \underline{+6x^2} & & \\
 & & -3x^2 & -4x & \\
 & & \underline{-3x^2} & \underline{-9x} & \\
 & & & 5x & +3 \\
 & & & \underline{5x} & \underline{+15} \\
 & & & & -12
\end{array}
$$

Since the remainder, $r(x) = -12$, is of a degree less than that of the divisor, the division is complete.

$2x^3 + 3x^2 - 4x + 3 = (x + 3)(2x^2 - 3x + 5) - 12$

b. Insert $0x$ in the function so that every term is present.

$$
\begin{array}{r|rrrr}
 & x^2 & +x & +2 & \\
\hline
x-2 & x^3 & -x^2 & +\mathbf{0x} & -4 \\
 & \underline{x^3} & \underline{-2x^2} & & \\
 & & x^2 & +0x & \\
 & & \underline{x^2} & \underline{-2x} & \\
 & & & 2x & -4 \\
 & & & \underline{2x} & \underline{-4} \\
 & & & & 0
\end{array}
$$

Since the remainder is 0, $x - 2$ is a factor of $x^3 - x^2 - 4$.

The other factor is $x^2 + x + 2$.

$x^3 - x^2 - 4 = (x-2)(x^2 + x + 2)$

EXAMPLE 4 Perform the following division and express the answer in the form $f(x) = d(x)q(x) + r(x)$.

$(3x^4 - 2x^3 + 4x^2 - 7x + 4) \div (x^2 - 3x + 1)$.

Solution

$$
\begin{array}{r|rrrrr}
 & & & 3x^2 & +7x & +22 \\
\hline
x^2-3x+1 & 3x^4 & -2x^3 & +4x^2 & -7x & +4 \\
 & \underline{3x^4} & \underline{-9x^3} & \underline{+3x^2} & & \\
 & & 7x^3 & +x^2 & -7x & \\
 & & \underline{7x^3} & \underline{-21x^2} & \underline{+7x} & \\
 & & & 22x^2 & -14x & +4 \\
 & & & \underline{22x^2} & \underline{-66x} & \underline{+22} \\
 & & & & 52x & -18
\end{array}
$$

Since the remainder, $r(x) = 52x - 18$, is of a lower degree than the divisor, $x^2 - 3x + 1$, the division is complete.

$3x^4 - 2x^3 + 4x^2 - 7x + 4 = (x^2 - 3x + 1)(3x^2 + 7x + 22) + (52x - 18)$

EXAMPLE 5 Determine the remainder when $9x^3 - 3x^2 - 4x + 2$ is divided by:

a. $3x - 2$

b. $x - \frac{2}{3}$

Solution

a.

$$\begin{array}{r}
3x^2 + x - \frac{2}{3} \\
3x - 2 \overline{)9x^3 - 3x^2 - 4x + 2} \\
\underline{9x^3 - 6x^2} \qquad\qquad \\
3x^2 - 4x \qquad \\
\underline{3x^2 - 2x} \qquad \\
-2x + 2 \\
\underline{-2x + \frac{4}{3}} \\
\frac{2}{3}
\end{array}$$

b.

$$\begin{array}{r}
9x^2 + 3x - 2 \\
x - \frac{2}{3} \overline{)9x^3 - 3x^2 - 4x + 2} \\
\underline{9x^3 - 6x^2} \qquad\qquad \\
3x^2 - 4x \qquad \\
\underline{3x^2 - 2x} \qquad \\
-2x + 2 \\
\underline{-2x + \frac{4}{3}} \\
\frac{2}{3}
\end{array}$$

The remainders are equal. Is this always true if a function is divided by $px + t$ and by $x + \frac{t}{p}$? Suppose that $f(x)$ divided by $d(x) = px + t$ produces quotient $q(x)$ and remainder $r(x)$. We can write $f(x) = (px + t)q(x) + r(x)$.

Now $f(x) = (px + t)q(x) + r(x)$

$$= p\left(x + \frac{t}{p}\right)q(x) + r(x)$$

$$= \left(x + \frac{t}{p}\right)[p \bullet q(x)] + r(x).$$

From this it is clear that division by $\left(x + \frac{t}{p}\right)$ produces a quotient greater by a factor p than that of division by $(px + t)$, but the remainders are the same.

Exercise 1.3

Part A

1. Perform each of the following divisions and express the result in the form **dividend = divisor × quotient + remainder.**

a. $17 \div 5$	b. $42 \div 7$	c. $73 \div 12$
d. $90 \div 6$	e. $103 \div 10$	f. $75 \div 15$

Communication

2. a. In Question 1 **a**, explain why 5 is not a factor of 17.

b. In Question 1 **b**, explain why 7 is a factor of 42.

c. In Questions 1 **d** and 1 **f**, what other divisor is a factor of the dividend?

Communication

3. Explain the division statement $f(x) = d(x)q(x) + r(x)$ in words.

Part **B**

Knowledge/Understanding

4. For $f(x) = (x - 2)(x^2 + 3x - 2) + 5$,

a. identify the linear divisor $d(x)$.

b. identify the quotient $q(x)$.

c. identify the remainder $r(x)$.

d. determine the dividend $f(x)$.

5. When a certain polynomial is divided by $x - 3$, its quotient is $x^2 - 5x - 7$ and its remainder is 5. What is the polynomial?

Application

6. When a certain polynomial is divided by $x^2 + x + 1$, its quotient is $x^2 - x + 1$ and its remainder is -1. What is the polynomial?

7. In each of the following, divide $f(x)$ by $d(x)$, obtaining quotient $q(x)$ and remainder r. Write your answers in the form $f(x) = d(x)q(x) + r(x)$.

a. $(x^3 - 3x^2 + x + 2) \div (x + 2)$

b. $(x^3 + 4x^2 - 3x - 2) \div (x - 1)$

c. $(2x^3 - 4x^2 - 3x + 5) \div (x - 3)$

d. $(3x^3 + x^2 - x - 6) \div (x + 1)$

e. $(3x^2 - 4) \div (x - 4)$

f. $(x^3 - 2x - 4) \div (x - 2)$

g. $(4x^3 + 6x^2 - 6x - 9) \div (2x + 3)$

h. $(3x^3 - 11x^2 + 21x - 7) \div (3x - 2)$

i. $(6x^3 + 4x^2 - 3x + 9) \div (3x - 2)$

j. $(3x^3 + 7x^2 + 5x + 1) \div (3x + 1)$

Communication

8. For the pairs of polynomials in Question 7, state whether the second is a factor of the first. If not, compare the degree of the remainder to the degree of the divisor. What do you observe?

Knowledge/Understanding

9. Perform the following divisions:

a. $(x^4 - x^3 + 2x^2 - 3x + 8) \div (x - 4)$

b. $(2x^4 - 3x^2 + 1) \div (x + 1)$

c. $(4x^3 + 32) \div (x + 2)$

d. $(x^5 - 1) \div (x - 1)$

10. One factor of $x^3 + 3x^2 - 16x + 12$ is $x - 2$. Find all other factors.

11. Divide $f(x) = x^3 + 2x^2 - 4x - 8$ by $x + 3$.

12. Divide $f(x) = x^4 + x^3 - x^2 - x$ by $d(x) = x^2 + 2x + 1$.

13. Divide $f(x) = x^4 - 5x^2 + 4$ by $d(x) = x^2 - 3x + 2$.

Thinking/Inquiry/ Problem Solving

14. In $f(x) = d(x)q(x) + r(x)$, what condition is necessary for $d(x)$ to be a factor of $f(x)$?

15. If $f(x) = d(x)q(x) + r(x)$ and $r(x) \neq 0$, given that the degree of $d(x)$ is 2, what are the possible degrees of $r(x)$?

Part C

Thinking/Inquiry/ Problem Solving

16. If x and y are natural numbers and $y \leq x$, then whole numbers q and r must exist such that $x = yq + r$.

a. What is the value of r if y is a factor of x?

b. If y is not a factor of x, what are the possible values of r if $y = 5$, $y = 7$, or $y = n$?

17. a. Divide $f(x) = x^3 + 4x^2 - 5x - 9$ by $x - 2$ and write your answer in the form $f(x) = (x - 2)q(x) + r_1$. Now divide $q(x)$ by $x + 1$ and write your answer in the form $q(x) = (x + 1)Q(x) + r_2$.

b. If $f(x)$ is divided by $(x - 2)(x + 1) = x^2 - x - 2$, is $Q(x)$ in part **a** the quotient obtained? Justify your answer.

c. When $f(x)$ is divided by $(x - 2)(x + 1)$, can the remainder be expressed in terms of r_1 and r_2?

Section 1.4 — The Remainder Theorem

With reference to polynomial functions, we can express the division algorithm as follows:

When a function $f(x)$ is divided by a divisor $d(x)$, producing a quotient $q(x)$ and a remainder $r(x)$, then $f(x) = d(x)q(x) + r(x)$, where the degree of $r(x)$ is less than the degree of $d(x)$.

Note that if the divisor is a linear function then the remainder must be a constant.

INVESTIGATION

The following investigation will illustrate an interesting way in which this relationship can be used.

1. a. For the function $f(x) = x^3 + x^2 - 7$, use long division to divide $(x^3 + x^2 - 7)$ by $(x - 2)$.
   b. What is the remainder?
   c. What is the value of $f(2)$?

2. a. Use long division to divide $(x^3 + 3x^2 - 2x + 1)$ by $(x + 1)$.
   b. What is the remainder?
   c. What is the value of $f(-1)$?

3. a. What was the relationship between $f(2)$ and the remainder in the first division?
   b. What was the relationship between $f(-1)$ and the remainder in the second division?
   c. Why do you think we chose the value 2 to use in Question 1 **c**?
   d. Why do you think we chose the value -1 to use in Question 2 **c**?

Based on these examples, complete the following statement:

When $f(x)$ is divided by $(x - 2)$, then the remainder $r(2) = f(\ )$.
When $f(x)$ is divided by $(x + 1)$, then the remainder $r(\ ) = f(\ )$.
When $f(x)$ is divided by $(x - a)$, then the remainder $r(\ ) = f(\ )$.

EXAMPLE 1 Show that for the function $f(x) = x^3 - x^2 - 4x - 2$, the value of $f(-2)$ is equal to the remainder obtained when $f(x)$ is divided by $(x + 2)$.

Solution

$$\begin{aligned} f(-2) &= (-2)^3 - (-2)^2 - 4(-2) - 2 \\ &= -8 - 4 + 8 - 2 \\ &= -6 \end{aligned}$$

$$\begin{array}{r@{}l} & x^2 - 3x + 2 \\ x + 2 & \overline{)x^3 - x^2 - 4x - 2} \\ & \underline{x^3 + 2x^2} \\ & \quad - 3x^2 - 4x \\ & \quad \underline{- 3x^2 - 6x} \\ & \qquad\qquad 2x - 2 \\ & \qquad\qquad \underline{2x + 4} \\ & \qquad\qquad\quad - 6 \end{array}$$

Since the remainder is -6, then the remainder equals $f(-2)$.

It appears that there is a relationship between the remainder and the value of the function. We now address this in general terms.

If the divisor is the linear expression $x - p$, we can write the division statement as $f(x) = (x - p)q(x) + r$. This equation is satisfied by all values of x. In particular, it is satisfied by $x = p$. Replacing x with p in the equation we get

$$\begin{aligned} f(p) &= (p - p)q(p) + r \\ &= (0)q(p) + r \\ &= r. \end{aligned}$$

This relationship between the dividend and the remainder is called the **Remainder Theorem**.

The Remainder Theorem If $f(x)$ is divided by $(x - p)$, giving a quotient $q(x)$ and a remainder r, then $r = f(p)$.

The Remainder Theorem allows us to determine the remainder in the division of polynomials without performing the actual division, which, as we will see, is a valuable thing to be able to do.

EXAMPLE 2 Find the remainder when $x^3 - 4x^2 + 5x - 1$ is divided by

a. $x - 2$ b. $x + 1$

Solution

Let $f(x) = x^3 - 4x^2 + 5x - 1$; therefore,

a. when $f(x)$ is divided by $x - 2$, the remainder is $f(2)$.

$$\begin{aligned} r &= f(2) \\ &= (2)^3 - 4(2)^2 + 5(2) - 1 \\ &= 1 \end{aligned}$$

b. when $f(x)$ is divided by $x + 1$, the remainder is $f(-1)$.

$$\begin{aligned} r &= f(-1) \\ &= (-1)^3 - 4(-1)^2 + 5(-1) - 1 \\ &= -11 \end{aligned}$$

What do we do if the divisor is not of the form $(x - p)$, but of the form $(kx - p)$? We have already seen that the remainder in dividing by $(kx - p)$ is the same as in dividing by $\left(x - \frac{p}{k}\right)$, so there is no difficulty. In this case, $r = f\left(\frac{p}{k}\right)$.

EXAMPLE 3 Find the remainder when $f(x) = x^3 - 4x^2 + 5x - 1$ is divided by $(2x - 3)$.

Solution

To determine the remainder, we write $2x - 3 = 2\left(x - \frac{3}{2}\right)$ and calculate $f\left(\frac{3}{2}\right)$.

The remainder is $r = f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^3 - 4\left(\frac{3}{2}\right)^2 + 5\left(\frac{3}{2}\right) - 1$

$$\begin{aligned} &= \frac{27}{8} - \frac{4 \times 9}{4} + \frac{15}{2} - 1 \\ &= \frac{7}{8}. \end{aligned}$$

EXAMPLE 4 When $x^3 + 3x^2 - kx + 10$ is divided by $x - 5$, the remainder is 15. Find the value of k.

Solution

Since $r = 15$ and $r = f(5)$, where $f(5) = 125 + 75 - 5k + 10$,

then $210 - 5k = 15$ (by the Remainder Theorem)

$$\begin{aligned} -5k &= -195 \\ k &= 39. \end{aligned}$$

We have noted that the remainder is always of a degree lower than that of the divisor. In the examples so far, the divisor was a linear function, so the remainder had to be a constant. In the next example, the divisor is a quadratic expression, so the remainder can be a linear expression.

EXAMPLE 5 Find the remainder when $x^4 + 2x^3 - 5x^2 + x + 3$ is divided by $(x + 2)(x - 1)$.

Solution 1 Using Long Division

Expand $(x + 2)(x - 1) = x^2 + x - 2$.

$$
\begin{array}{r l}
 & \phantom{x^4 + 2x^3 -{}} x^2 + x - 4 \\
x^2 + x - 2 \,\big) & \overline{x^4 + 2x^3 - 5x^2 + x + 3} \\
 & \underline{x^4 + x^3 - 2x^2} \\
 & \phantom{x^4 +{}} x^3 - 3x^2 + x \\
 & \phantom{x^4 +{}} \underline{x^3 + x^2 - 2x} \\
 & -4x^2 + 3x + 3 \\
 & \underline{-4x^2 - 4x + 8} \\
 & \phantom{x^4 + x^3 - 4x^2 +{}} 7x - 5
\end{array}
$$

The remainder is $7x - 5$.

Solution 2 Using the Remainder Theorem

We have $f(x) = x^4 + 2x^3 - 5x^2 + x + 3$. ①

Then $f(x) = (x + 2)(x - 1)q(x) + r(x)$ where $r(x)$ is at most a linear expression.

Let $\quad r(x) = Ax + B$.

Now $\quad f(x) = (x + 2)(x - 1)q(x) + (Ax + B)$. ②

From ② $\quad f(1) = (3)(0)q(1) + A + B$

$\quad = A + B$.

From ① $\quad f(1) = 1^4 + 2(1)^3 - 5(1)^2 + 1 + 3 = 2$.

So $\quad A + B = 2$. ③

Similarly $\quad f(-2) = (0)(-3)8(x) + (-2A + B) = -2A + B$

and $\quad f(-2) = (-2)^4 + 2(-2)^3 - 5(-2)^2 - 2 + 3 = -19$

so $\quad -2A + B = -19$. ④

We solve equations ③ and ④ for A and B.

$$A + B = 2$$
$$-2A + B = -19$$

Subtracting $3A = 21$

$$A = 7 \text{ and } B = -5$$

Since $r(x) = Ax + B$, the remainder is $7x - 5$.

Exercise 1.4

Part A

Communication

1. Explain how you determine the remainder when $x^3 + 4x^2 - 2x - 5$ is divided by $x - 1$.

2. What is the remainder when $x^3 - 4x^2 + 2x - 6$ is divided by

a. $x - 2$ b. $x + 1$ c. $2x - 1$ d. $2x + 3$

3. Determine the remainder in each of the following:

a. $(x^2 + 3) \div (x - 3)$
b. $(x^3 + x^2 - x + 2) \div (x - 1)$
c. $(2x^3 + 4x - 1) \div (x + 2)$
d. $(3x^4 - 2) \div (x + 1)$
e. $(x^4 - x^2 + 5) \div (x - 2)$
f. $(-2x^4 + 3x^2 - x + 2) \div (x + 2)$

Part B

Knowledge/Understanding

4. Determine the remainder in each of the following using the Remainder Theorem:

a. $(x^3 - 2x^2 + 3x + 4) \div (x + 1)$
b. $(x^4 - x^3 + x^2 - 3x + 4) \div (x - 3)$
c. $(x^3 + 3x^2 - 7) \div (x - 2)$
d. $(x^5 - 1) \div (x - 1)$
e. $(6x^2 - 10x + 7) \div (3x + 1)$
f. $(4x^3 + 9x - 10) \div (2x - 1)$
g. $(x^3 + 3x^2 - x - 2) \div (x + 3)$
h. $(3x^5 - 5x^2 + 4x + 1) \div (x - 1)$

Application

5. Determine the value of k in each of the following:

a. When $x^3 + kx^2 + 2x - 3$ is divided by $x + 2$, the remainder is 1.
b. When $x^4 - kx^3 - 2x^2 + x + 4$ is divided by $x - 3$, the remainder is 16.
c. When $2x^3 - 3x^2 + kx - 1$ is divided by $2x - 1$, the remainder is 1.

Thinking/Inquiry/Problem Solving

6. If $f(x) = mx^3 + gx^2 - x + 3$ is divided by $x + 1$, the remainder is 3. If $f(x)$ is divided by $x + 2$, the remainder is -7. What are the values of m and g?

7. If $f(x) = mx^3 + gx^2 - x + 3$ is divided by $x - 1$, the remainder is 3. If $f(x)$ is divided by $x + 3$, the remainder is -1. What are the values of m and g?

Part C

8. Determine the remainder when $(x^3 + 3x^2 - x - 2)$ is divided by $(x + 3)(x + 5)$.

9. Determine the remainder when $(3x^5 - 5x^2 + 4x + 1)$ is divided by $(x - 1)(x + 2)$.

Thinking/Inquiry/ Problem Solving

10. When $x + 2$ is divided into $f(x)$, the remainder is 3. Determine the remainder when $x + 2$ is divided into each of the following:
   a. $f(x) + 1$
   b. $f(x) + x + 2$
   c. $f(x) + (4x + 7)$
   d. $2f(x) - 7$
   e. $[f(x)]^2$

11. If $f(x) = (x + 5)q(x) + (x + 3)$, what is the first multiple of $(x + 5)$ greater than $f(x)$?

12. The expression $x^4 + x^2 + 1$ cannot be factored using known techniques. However, by adding and subtracting x^2, we obtain $x^4 + 2x^2 + 1 - x^2$.

$$\begin{aligned}\text{Therefore, } x^4 + 2x^2 + 1 - x^2 &= (x^2 + 1)^2 - x^2 \\ &= (x^2 + x + 1)(x^2 - x + 1).\end{aligned}$$

Use this approach to factor each of the following:
   a. $x^4 + 5x^2 + 9$
   b. $9y^4 + 8y^2 + 4$
   c. $x^4 + 6x^2 + 25$
   d. $4x^4 + 8x^2 + 9$

Key Concepts Review

After your work in this chapter on Polynomial Functions, you should be familiar with the following concepts:

Factoring Types

You should be able to identify and simplify expressions of the following types:

- common
- trinomial
- grouping
- difference of squares

Sketching Polynomial Functions

- Make use of the relationships between x-intercepts and the roots of the corresponding equation to sketch the graph of functions.

Division of Polynomials

Remainder Theorem

- If $f(x)$ is divided by $(x - a)$, giving a quotient $q(x)$ and a remainder r, then $r = f(a)$.

Polynomial Functions from Data

- The first differences of a linear function are constant.
- The second differences of a quadratic function are constant.
- The third differences of a cubic function are constant.

Career Link wrap-up investigate and apply

CHAPTER 1: MODELS FOR WATER FLOW RATES

1. Using the data presented in the Career Link, develop and utilize a polynomial mathematical model of the flow-rate and time relationship [$Q = f(t)$] by

   a. determining the degree of the polynomial, then using the graphing calculator to obtain an algebraic model for $Q = f(t)$ with the appropriate polynomial regression function.

   b. using the graphing calculator to determine the peak flow. When does this occur? Is this a reasonable time for a peak daily flow? Explain.

   c. determining an algebraic model for the velocity [$V(t)$] of the water in the pipe (metres per hour) leaving the water plant if the cross-sectional area [$A(t)$] of the pipe changes over time with the relationship:

   $$A(t) = 0.1t + 0.4$$

   where $A(t)$ is cross-sectional area in square metres, t is time in hours, and $Q(t) = A(t)\ V(t)$.

   d. verifying that your model in part **c** is correct using the graphing calculator. Explain how you did this.

2. Water travelling at high velocities can cause damage due to excessive forces at bends (elbows) in pipe networks. If the maximum allowable velocity in this specific pipe is 2.5 m/s, will the pipe be damaged at the peak flow rate?

Review Exercise

1. Draw a sketch of each of the following without using your graphing calculator.

   a. $y = (x - 2)(x + 3)$
   b. $y = -(x + 3)^2 + 1$
   c. $y = x(x - 1)(x - 3)$
   d. $y = (x + 2)(x - 4)(x - 2)$
   e. $y = -(x - 2)^3$
   f. $y = -(x + 4)(x - 1)(x + 3)$
   g. $y = (x + 2)^2 (x - 4)$
   h. $y = (x - 2)^2(x + 1)^2$
   i. $y = -x^2(x - 3)(x + 2)$
   j. $y = (x - 4)(x + 1)(x + 2)(x - 3)$
   k. $y = (x - 2)^3(x + 3)$
   l. $y = -x(x + 2)(x - 3)$

2. In each of the following, you are given a set of points that lie on the graph of a polynomial function. If possible, determine the equation of the function.

   a. $(-1, -27), (0, -11), (1, -5), (2, -3), (3, 1), (4, 13)$
   b. $(0, 4), (1, 15), (2, 32), (3, 67), (4, 132), (5, 239)$
   c. $(1, -9), (2, -31), (3, -31), (4, 51), (5, 299), (6, 821)$
   d. $(1, 1), (2, 2), (3, 5), (4, 16)$
   e. $(-2, 75), (-1, -11), (0, -21), (1, -27), (2, -53)$

3. Perform the following divisions:

   a. $(x^3 - 2x^2 + 3x - 1) \div (x - 3)$
   b. $(2x^3 + 5x + 4) \div (x + 2)$
   c. $(4x^3 + 8x^2 - x + 1) \div (2x + 1)$
   d. $(x^4 - 4x^3 + 3x^2 - 3) \div (x^2 + x - 2)$

4. Without using long division, determine the remainder when

   a. $(x^2 - x + 1)$ is divided by $(x - 2)$.
   b. $(x^3 + 4x^2 - 2)$ is divided by $(x + 1)$.
   c. $(x^3 - 5x^2 + 2x - 1)$ is divided by $(x + 2)$.
   d. $(x^4 - 3x^2 + 2x + 3)$ is divided by $(x + 1)$.
   e. $(3x^3 + x + 2)$ is divided by $(3x - 1)$.

5. Divide each polynomial by the factor given, then express each polynomial in factored form.

   a. $x^3 + 2x^2 - x - 2$, given $x - 1$ is a factor.

   b. $x^3 - 3x^2 - x + 3$, given $x - 3$ is a factor.

   c. $6x^3 + 31x^2 + 25x - 12$, given $2x + 3$ is a factor.

6. a. When $x^3 - 3kx^2 + x + 5$ is divided by $x - 2$, the remainder is 9. Find the value of k.

   b. When $rx^3 + gx^2 + 4x + 1$ is divided by $x - 1$, the remainder is 12. When it is divided by $x + 3$, the remainder is -20. Find the values of r and g.

Chapter 1 Test

Achievement Category	Questions
Knowledge/Understanding	1, 3, 5, 7b
Thinking/Inquiry/Problem Solving	8
Communication	4
Application	2, 6, 7a, 9

1. Factor each of the following:
   a. $18x^2 - 50y^2$
   b. $pm^3 + m^2 + pm + 1$
   c. $12x^2 - 26x + 12$
   d. $x^2 + 6y - y^2 - 9$

2. Without using a graphing calculator, sketch the graph of
   a. $y = (x + 2)(x - 1)(x - 3)$
   b. $y = x^2(x - 2)$

3. Find the quotient and remainder when
   a. $x^3 - 5x^2 + 6x - 4$ is divided by $x + 2$.
   b. $(x^3 - 6x + 2)$ is divided by $(x - 3)$.

4. Since $f(1) = 0$ for $f(x) = 4x^3 - 6x + 2$, do you think $(x - 1)$ is a factor of $f(x) = 4x^3 - 6x + 2$? Explain.

5. Without using long division, find the remainder when $(x^3 - 6x^2 + 5x + 2)$ is divided by $(x + 2)$.

6. Find the value of k if there is a remainder of 7 when $x^3 - 3x^2 + 4x + k$ is divided by $(x - 2)$.

7. a. Do $(1, -1)$, $(2, -1)$, $(3, 1)$, $(4, 5)$ lie on the graph of a quadratic function?
   b. Use your graphing calculator to find the simplest polynomial function that contains the following points: $(1, -4)$, $(2, 6)$, $(3, 34)$, $(4, 92)$.

8. When $x^3 + cx + d$ is divided by $x + 1$, the remainder is 3, and when it is divided by $x - 2$, the remainder is -3. Determine the values of c and d.

9. One factor of $x^3 - 2x^2 - 9x + 18$ is $x - 2$. Determine the other factors.